

ADE-quiver theories and Mirror Symmetry*

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We show that the Higgs branch of a four-dimensional Yang-Mills theory, with gauge and matter content summarised by an *ADE* quiver diagram, is identical to the generalised Coulomb branch of a four-dimensional superconformal strongly coupled gauge theory with *ADE* global symmetry. This equivalence suggests the existence of a mirror symmetry between the quiver theories and the strongly coupled theories.

1. INTRODUCTION

The 3D mirror symmetry between the Higgs and the Coulomb branch described in [4] seems to have a 4D counterpart in a mirror symmetry between the Higgs branch of an *ADE* quiver gauge theory and the (generalized) Coulomb branch of a Seiberg-Witten (SW) theory with *ADE* global symmetry. This symmetry was suggested by the results of [1], where the algebraic curve⁷ for the *ADE* series of four dimensional *ALE* manifolds was related to the description of these manifolds as hyperkähler quotients [8,9]. Inclusion of Fayet-Iliopolous (FI) parameters in the quotient leads to the deformed *ADE*-curves, and the curve for E_6 surprisingly turned out to be identical to the SW curve of a superconformal theory with

E_6 global symmetry described in [11–13]. This agreement between the curves was seen when the FI parameters were substituted by the Casimirs of the group E_6 . The same agreement has since been found for E_7 in [2] and for E_8 in [3].

There is thus a strong case for the proposed mirror symmetry and hence for a duality similar to that in three dimensions. This result is potentially very useful for at least two reasons. Firstly, the strongly coupled superconformal theories with E_n global symmetry have no Lagrangian description, whereas their mirror images do. Certain aspects of the theories that are better studied in a Lagrangian formulation may thus be investigated in the mirror theory. Secondly, since the Coulomb branch receives quantum corrections but the Higgs branch does not, one consequence of the mirror symmetry is that quantum effects in one theory arise classically in the dual theory, and vice versa.

There is also an advantage in having found the relation between the deformation parameters of the algebraic curves and the FI parameters. Assume that one is contemplating a Hanany-Witten (HW) picture of NS5-branes with D4-branes end-

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⁷We are not being very careful with the notation. "Curve" should really be *variety*, but we hope that this will not cause confusion.

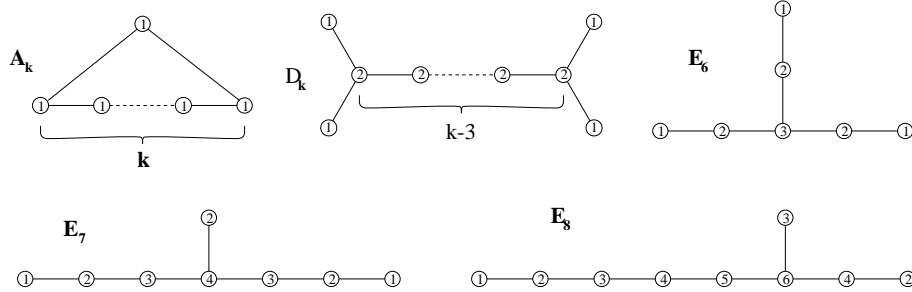


Figure 1. The extended Dynkin diagrams for the ADE quiver theories.

ing on them as a candidate for the IIA T-dual of D3-branes on an E_n singularity in IIB. As described in [3], and in section 5 below, moving the NS5-branes in the HW picture corresponds to blowing up the singularity in the dual picture, the FI parameters giving the position of the NS5-branes.

Since we have the relation of the FI parameters to the parameters governing the deformation of the algebraic curve, one may now check that the possible motions on the HW side (allowed by the particular geometry suggested) correspond to the known allowed deformations on the IIB side. In other words, our results may serve as a guideline when trying to find a HW picture.

Below we describe the derivation of the results, in particular the “bug calculus” that made the derivation technically feasible.

2. The ADE -series

The quiver theories [7,6] are $\mathcal{N} = 2$ supersymmetric gauge theories that may be characterized by the extended Dynkin diagrams of the ADE -series (*quiver diagrams*), as depicted in figure 1. Here the gauge group is $U(N_1) \times \dots \times U(N_k)$, and the i 'th node, labelled by N_i in the Dynkin diagram, corresponds to a factor $U(N_i)$ in the gauge group. Moreover, each link between two nodes i, k corresponds to a hypermultiplet in the (N_i, \bar{N}_k) representation. The Dynkin diagram thus sums up both the gauge group and the matter content of the quiver theory. These theories may be constructed as the worldvolume theory of

D3-branes probing an orbifold singularity.

There is also a closely related point of view where the Dynkin diagram represents a hyperkähler quotient by the above gauge group and the hypermultiplets coordinatize a $4D$ ALE -space. It is this latter point of view which we take as the starting point for our investigations.

The hyperkähler quotient construction [8,9] starts from a $\mathcal{N} = 2$ supersymmetric nonlinear σ model coupled to an $\mathcal{N} = 2$ vector multiplet (including FI terms). In $\mathcal{N} = 1$ language the hypermultiplets and vectormultiplets involved are given by (z_+, z_-) and V, S , respectively, where z_{\pm} and S are $\mathcal{N} = 1$ chiral superfields and V is an $\mathcal{N} = 1$ vector superfield. With the above gauge groups, the quotient found by integrating out (V, S) produces a new σ -model with an ALE -space as target space. We shall be particularly interested in the so called moment map constraints, i.e., the equations that result from integrating out S :

Classification	Polynomial	Deformations
A_k	$XY - Z^{k+1}$	$1, \dots, Z^{k-1}$
D_k	$X^2 + Y^2Z - Z^{k-1}$	$1, Y, Z, \dots, Z^{k-2}$
E_6	$X^2 + Y^3 - Z^4$	$1, Y, Z, YZ, Z^2, YZ^2$
E_7	$X^2 + Y^3 + YZ^3$	$1, Y, Y^2, Z, YZ, Z^2, Y^2Z$
E_8	$X^2 + Y^3 + Z^5$	$1, Y, Z, YZ, Z^2, Z^3, YZ^2, YZ^3$

Table 2

$$\begin{aligned} z_+ T_A z_- &= 0 & A \notin \text{any } U(1) \text{ factor} \\ &= b_A & A \in \text{any } U(1) \text{ factor,} \end{aligned} \quad (1)$$

where b_A are FI parameters and A is a group index. Turning off the FI terms results in the orbifold limit of the *ALE*-space, and conversely non-zero FI terms correspond to resolutions of the orbifold.

The *ALE*-spaces, classified by the *ADE* series, also have a description in terms of an algebraic curve in \mathbb{C}^3 [10]. Here the resolution of the orbifold corresponds to certain allowed deformations, as listed in Table 2.

Our goal is to find the relation between the FI parameters and the deformations of the algebraic curves. The strategy is to form gauge group invariants from the hypermultiplets and identify those invariants with the coordinates X, Y and Z of the curves in Table 2. All the calculations should be done taking the constraints (1) into account. Algebraically finding the curve with non-zero FI terms is rather a formidable task for most of the models. It is considerably simplified, however, by use of a “bug calculus” [1], which we now describe.

3. Bug calculus

We start from the Dynkin diagrams in Table 1 and associate a FI parameter b_i with the i 'th node. We also need to keep track of orientation; an arrow from the fundamental towards the anti-fundamental representation indicates the way the chiral field in the hypermultiplet transforms⁸. It is then possible to form matrices from the matter fields and depict them graphically. E.g., the holomorphic constraints (1) can be represented in bug calculus, each gauge group (*i.e.*, node) having its own constraint. For an “endpoint” the constraint is shown in figure 2a and for a node in a chain the constraint is shown in figure 2b. For nodes connecting more than two links, the constraints generalize as indicated in figure 2c and d. When manipulating the bugs, the moves are

⁸A change of direction of an arrow only affects the final result by a change of sign of the corresponding FI parameter.

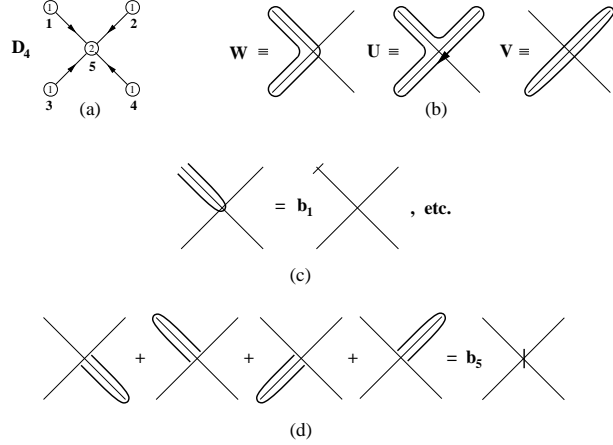


Figure 3. Diagrammatic representation of the D_4 invariants and moment map constraints.

dictated by the moment map constraints. Using them, one immediately finds that some traces of matrices reduce to polynomials in b_i and eventually one is left with a set of non-reducible invariants. Additional use of the constraints then leads to a relation between (products of) these variables, which is the candidate for the curve. For D_4 , the Dynkin diagram, the invariants and the constraint are given in figures 3a, 3b and 3c-d, respectively [1]. Some of the moves are described in figure 4. Figure 4a expresses a four-link diagram in terms of the basic four-link diagrams W and V . Figures 4b and 4c relate U to its orientation reversed image. Figure 4d yields the algebraic curve in diagrammatic form. substituting 4a-c and similar relations into 4d we find the curve

$$\begin{aligned} U^2 + U[(b_4 - b_1)V + (b_4 - b_2)W + a_1] \\ - W^2V - WV^2 + a_2WV = 0, \end{aligned} \quad (2)$$

where a_1 and a_2 are polynomials in the FI parameters. To find the standard form of the D_4 curve, as given in Table 1, we have to shift the variables according to

$$\begin{aligned} U &= \frac{1}{2}[X + (b_1 - b_4)V + (b_2 - b_4)W - a_1] \\ V &= \frac{1}{2}[Y - W + a_2 - \frac{1}{2}(b_1 - b_4)(b_2 - \frac{1}{2}(b_1 + b_4))] \\ W &= -Z - \frac{1}{4}(b_1 - b_4)^2. \end{aligned} \quad (3)$$

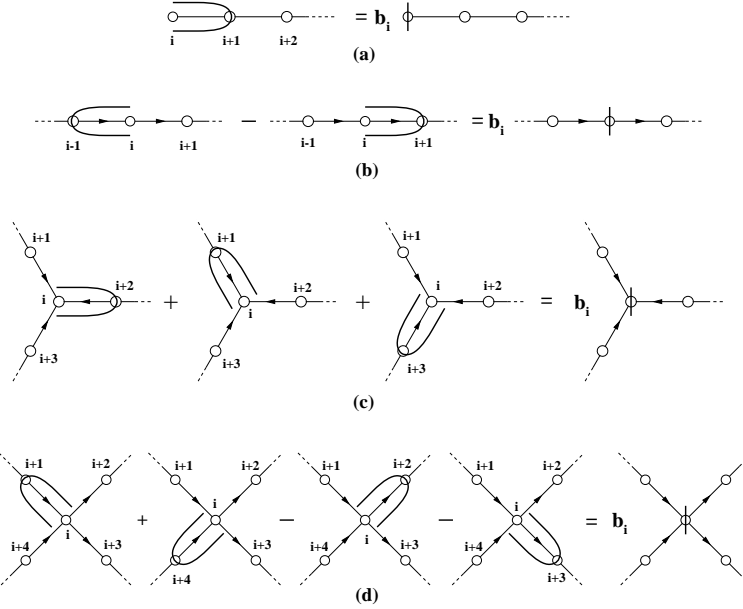


Figure 2. The bug calculus. b_i is the fayet-Iliopolous parameter associated with the i 'th node, and a vertical bar through the i 'th node represents a $U(N_i)$ Kronecker- δ

The result is thus an explicit expression for the deformations of the algebraic curve with (functions of) the FI parameters as coefficients in the curve. Modulo the technicalities such as a fair amount of bug calculus, use of certain Schouten identities etc, this sums up the procedure for the A_k and D_k series.

For brevity we do not display the full result here, but note that the quantities that enter in the expression for the deformations in both these series are related to the weights of the fundamental representations of the respective Lie algebras, if we think of each FI parameter as the simple root associated to its node in the Dynkin diagram. This observation becomes crucial when we turn to the E -series, both for organizing the results in a comprehensive way and for finding the relation to the SW models.

4. The E_n series

Although more involved, the procedure for deriving the deformations of the E_n -curves follows

the lines described in section 3 [1]-[3]. The key to understanding the initially not very illuminating results is to first find an expression for the Casimirs of the Lie algebras in terms of the FI parameters, and then invert this. The final expressions for the deformed curves in terms of these Casimirs are manageable and in fact known; they are the SW curves for the superconformal “fixed point” theories described in [11–13].

The i 'th Casimir P_i can be found as the coefficient of x^{d_n-i} in the polynomial

$$\det(x - v \cdot H), \quad (4)$$

where d_n is the dimension of the fundamental representation of E_n and v is an arbitrary vector in the Cartan subspace. The matrix $v \cdot H$ is given in terms of the weights λ of this representation as $v \cdot H = \text{diag}(v \cdot \lambda_1 \dots v \cdot \lambda_d)$. Using that each FI parameter b_i can be thought of as the scalar product between v and its corresponding root we rewrite the weights in terms of the FI parameters. This yields the relation between the P_i 's and the b_i 's we were looking for.

When comparing our results to the SW curves, the most immediate comparison is with [13] where the curves are given in terms of the Casimirs. On the other hand, the expressions in [11,12] are in terms of mass-parameters m_i and the relation $P_i = P_i(m)$ thus gives us an interpretation of the b_i 's in terms of mass parameters.

While the above description of the derivation gives the principles of the procedure, there are many technical obstacles, most notably in the E_8 calculation [3]. In fact, although the bug calculus is very efficient (many pages of algebra are replaced by a few figures) it was not by itself enough to allow us to perform the E_8 calculation. Firstly we had to perform the calculations using a computer program (MAPLE), and secondly we could not do all of the comparison to the SW curve explicitly. To deal with some of the highest order terms (e.g., a polynomial of order 30 in eight variables) we had to resort to numerical methods: inserting random prime numbers we found that also the terms most difficult to compare agreed. It is thus clear that the *ADE* quiver theories away from the orbifold limit have deformed algebraic curves that are identical to the SW curves of certain superconformal theories with the corresponding global symmetries.

The interpretation of this fact, suggested in [1], substantiated in [2] and further discussed in [3], is the existence of a mirror symmetry between the Higgs branch of the quiver theories and the Coulomb branch of the SW theories similar to that which exists for $3D$ gauge theories [4]. A problem here, though, is that the Higgs branch is a hyperkähler manifold, whereas the Coulomb branch in $4D$ is not (in general). This is overcome by the proposal in [2] that what is involved is the *generalized* Coulomb branch, defined to be the $4D$ elliptically fibered space obtained by fibering the SW torus over the usual Coulomb branch.

5. Geometrical interpretation of b_i

We close this presentation with a section quoted from [3] on the geometrical meaning of the FI terms.

The A_{n-1} quiver theories can be viewed as the world-volume theory of $D3$ -branes on a $\mathbb{C}^2/\mathbb{Z}_n$

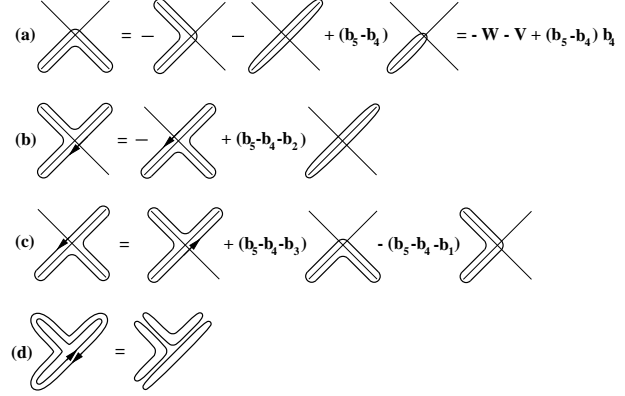


Figure 4. Some typical moves for the D_4 example.

orbifold singularity in type IIB string theory. This picture is T-dual to a picture of type IIA string theory in a background of $D4$ -branes stretching between $NS5$ -branes [14]. This dual picture, the Hanany-Witten (HW) picture [5], provides an intuitive geometric interpretation of blow-ups of A_{n-1} type singularities. An analogous picture exists for D_n type singularities [15,16], and it seems plausible that there are generalizations also to E_6 , E_7 and E_8 . In this section, we analyze the HW picture for the $\mathbb{C}^2/\mathbb{Z}_n$ case along the lines of [14] (see also [17,18]); in particular we clarify the role of the Fayet-Iliopoulos terms.

Starting from the type IIB string theory configuration (\times means the object is extended in that direction, and $-$ means it is point-like)

	0	1	2	3	4	5	6	7	8	9
sing	\times	\times	\times	\times	\times	\times	$-$	$-$	$-$	$-$
D3	\times	\times	\times	\times	$-$	$-$	$-$	$-$	$-$	$-$

we T-dualize along the 6-direction to get

	0	1	2	3	4	5	6	7	8	9
NS5	\times	\times	\times	\times	\times	\times	$-$	$-$	$-$	$-$
D4	\times	\times	\times	\times	$-$	$-$	\times	$-$	$-$	$-$

in type IIA string theory in flat space-time. There are n $NS5$ -branes, which all coincide in the 789 directions, but not necessarily in the 6-direction.

Between them D4-branes are suspended, which are the T-duals of the IIB D3-branes. The rotational symmetry $SO(3) \simeq SU(2)$ of the 789 coordinates translates into the $SU(2)_R$ symmetry of the gauge theory living on the D4-branes. The hypermultiplets arise from fundamental strings stretching across the NS5-branes, between neighboring D4-branes.

Resolving singularities in the IIB picture corresponds to separating NS5-branes along the 789 directions in the IIA picture. By an $SU(2)$ rotation we can always pick the direction of displacement to be x^7 . Note that such a displacement breaks the 789 rotational symmetry; that is, blowing up a singularity breaks the $SU(2)_R$ symmetry. If we move some of the NS5-branes in this way, with the D4-branes still stuck to them, and then T-dualize along x^6 again, we do not regain the D3-brane picture. Rather, the now tilted D4-branes dualize to a set of D5-branes (with nonzero B-field) with their 67 world-volume coordinates wrapped on 2-cycles. Shrinking these 2-cycles to zero size, each of the wrapped D5-branes is a fractional D3-brane, which cannot move away from the singularity. Thus a fractional D3-brane corresponds to a D4-brane whose ends are stuck on NS5-branes.

To move a fractional D3-brane, or, equivalently, a wrapped D5-brane, along the 6789 directions, we need to add $n - 1$ images (under \mathbf{I}_n), all associated with a 2-cycle each. The sum of the full set of 2-cycles is homologically trivial and can be shrunk to zero size. Then the collection of wrapped D5-branes will look like a single D3-brane that can move around freely in the orbifold. This procedure corresponds in the HW picture to starting out with a single D4-brane stretching between two of the n NS5-branes, and wanting to move the D4-brane (in the 7-direction, say) away from the NS5-branes, detaching its ends. In order not to violate the boundary conditions of the D4-brane, we then need to put one D4-brane between each unconnected pair of NS5-branes and join them at the ends. We then get a total of n D4-branes forming a single brane winding once around the periodic 6-direction. The D4-brane may now be lifted off the NS5-branes and can move freely, corresponding to the free D3-brane in the T-dual picture.

We may also gain some insight concerning the role played by the FI parameters in the HW picture, from the world-volume theory of a wrapped D5-brane on the orbifold singularity. Consider such a brane living in the 012367 directions, with its 67 world-volume coordinates wrapped on a 2-cycle Ω_k . The Born-Infeld and Chern-Simons terms in the world-volume action are, schematically, [7]

$$\begin{aligned} I_{D5} = & \int d^6x \sqrt{\det(g + \mathcal{F})} + \mu \int C^{(6)} \\ & + \mu \int C^{(4)} \wedge \mathcal{F} + \mu \int C^{(2)} \wedge \mathcal{F} \wedge \mathcal{F} \\ & + \mu \int C^{(0)} \wedge \mathcal{F} \wedge \mathcal{F} \wedge \mathcal{F}, \end{aligned} \quad (5)$$

where g is the metric on the world-volume, $C^{(p)}$ is the R-R p -form, μ is a constant, and $\mathcal{F} = F^{(2)} + B^{NS}$ where the 2-form $F^{(2)}$ is the field strength of the gauge field on the brane and B^{NS} is the NS-NS 2-form on the brane. Dimensional reduction to the 0123 directions, by integrating over the 2-cycle, puts the first term of (5) on the form

$$\int_{\Omega_k} d^2x \sqrt{\det(g_2 + \mathcal{F}_2)} \int d^4x \sqrt{\det(g_4 + \mathcal{F}_4)}, \quad (6)$$

where $\mathcal{F}_2 = C^{(2)} + B^{NS}$, g_2 is the metric on the 67 directions, and g_4 is the metric on the 0123 directions. Expanding (6) we obtain the coupling constant g_k^{-2} in four dimensions as the coefficient of $\int d^4x F_{\mu\nu} F^{\mu\nu}$. It is just the factor on the left in (6), which we can write as

$$g_k^{-2} = \left| \int_{\Omega_k} (B^{NS} + iJ) \right|. \quad (7)$$

In the HW picture the coupling constant of the four dimensional theory is proportional to the length of the D4-brane in the 6-direction. Hence (7) measures the total distance between two NS5-branes between which the D4-brane is suspended. Furthermore, since the distance between the NS5-branes in the isometry direction (in our case x^6) is given by the flux of the B^{NS} field on the corresponding cycle, we have to interpret $\int_{\Omega_k} J$ as the position of the NS5-branes in a direction orthogonal to that, let us choose x^7 . Movement of the NS5-branes in the remaining directions x^8 and x^9

now corresponds to turning on the $SU(2)_R$ partners of the Kähler form.

The integral of J over a 2-cycle is also, by definition, a Fayet-Iliopoulos term. A hyperkähler manifold has an $SU(2)$ manifold of possible complex structures. Choosing a complex structure we can define the Kähler form J as ω^1 , and the holomorphic 2-form as $\omega^2 + i\omega^3$. These three 2-forms rotate into each other under $SU(2)_R$ transformations, corresponding to choosing a different complex structure. The k :th triplet of FI terms is defined by the period of $\vec{\omega} = (\omega^1, \omega^2, \omega^3)$ (and hence also transforms as a triplet under $SU(2)_R$), as

$$\vec{\zeta}_k \equiv \int_{\Omega_k} \vec{\omega}.$$

Hence

$$\zeta_k^R = \int_{\Omega_k} J,$$

where ζ_k^R is the real component of the triplet of FI terms $\vec{\zeta}_k = (\zeta_k^R, \zeta_k^C, \overline{\zeta_k^C})$.

Another way to obtain the FI terms of the four-dimensional Yang-Mills theory is via dimensional reduction and supersymmetrization of the D5-brane world-volume theory [7]. The third term of (5) can be rewritten as

$$\int d^6x (A_\mu - \partial_\mu c^{(0)})^2,$$

where $c^{(0)}$ is the Hodge dual potential of $C^{(4)}$ in six dimensions. After integration over the k :th 2-cycle we supersymmetrize this to

$$\int d^4x d^4\theta (\mathbf{C}_k - \overline{\mathbf{C}}_k - \mathbf{V})^2,$$

where \mathbf{C}_k is a chiral superfield whose complex scalar component is $c^{(0)} + i\zeta_k^R$, and \mathbf{V} is the vector superfield containing A_μ . Here the imaginary part ζ_k^R of the scalar component is the real FI term in four dimensions, and we see that it arises as the superpartner of $c^{(0)}$.

6. Conclusions

As mentioned in the introduction, the mirror symmetry found is useful because it relates quantum and classical regimes as well as theories without a Lagrangian formulation to theories with

such a formulation. Also mentioned is that the geometrical interpretation of the FI terms may serve as a guide-line for finding dual HW pictures of the $D3$ -branes on the E_n singularities.

In [1] it is shown that higher dimensional hyperkähler quotients may also be related to quiver diagrams, although the connection to the simple Lie algebra classification is lost. In particular, several four (complex) dimensional spaces were constructed. This opens up the possibility of a systematic investigation of these spaces, perhaps leading to an eventual classification. The physical relevance of such spaces is not obvious, but perhaps they have a place in an F -theory picture.

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